shows that use of the Hertz solution results in substantial errors for large contact domains.

REFERENCES

1. Bondareva, V. F., Contact problems for an elastic sphere. PMM Vol. 35, № 1, 1971.
2. Abramian, B. L., Arutiunian, N. Kh. and Babloian, A. A., On two contact problems for an elastic sphere. PMM Vol. 28, № 4, 1964.
3. Karpenko, V. A., Axisymmetric contact problem for an elastic sphere. Izv. Akad. Nauk ArmSSR, Mekhanika, Vol. 24, № 4, 1971.
4. Bondareva, V. F., On the effect of an axisymmetric normal loading on an elastic sphere. PMM Vol. 33, № 6, 1969.
5. Aleksandrov, V. M., On the approximate solution of a certain type of integral equations. PMM Vol. 26, № 5, 1962.

Translated by M. D. F.

UDC 539.376

NEW EULER STABILITY CRITERION FOR A VISCOELASTIC ROD

PMM Vol. 40, № 4, 1976, pp. 766-768

E. I.-G. GOL'DENGERSHEL'
(Moscow)

(Received February 6, 1975)

A detailed exposition of the mechanical results announced in [1] is given below.

Let us suppose that a thin viscoelastic variable-section rod of finite length l is subjected to weak bending, under the action of longitudinal compressive force P, and under the influence of a slowly varying external transverse load $p(x, t)$.

Then the deflection $y(x, t)$ of the rod axis is described by the following boundary value problem [2 — 4]:

$$-\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) - P\frac{\partial^2 y}{\partial x^2} - P\int_0^t K(t, \tau)\frac{\partial^2 y}{\partial x^2}d\tau = \tag{1}$$

$$-p(x, t) - \int_0^t K(t, \tau)\, p(x, \tau)\, d\tau, \quad 0 \leqslant x \leqslant l, \quad 0 \leqslant t < \infty$$

$$U_i[y] = 0, \quad i = 1, 2, 3, 4 \tag{2}$$

Here the notation introduced in [1], and the conditions imposed on the moment of inertia $I(x)$, the creep kernel $K(t, \tau)$ and the left sides of the boundary conditions $U[y]$, are retained. We also proceed from the definition of Euler stability and the critical value of the force P contained in [1]. (Another approach to this question is contained in [5]).

The purpose of this paper is to obtain a lower bound and an exact formula for the critical value of the force P under substantially more general conditions than in [4]. Theorem 1 from [1] on the spectrum of the Volterra operator V

$$(Vf)(t) = \int_0^t K(t, \tau) f(\tau)\, d\tau, \quad 0 \leqslant t < \infty \tag{3}$$

in the Banach space $M_{\langle\alpha(t)\rangle}$ and its subspaces $\Lambda_{\langle\alpha(t)\rangle}$ and $Z_{\langle\alpha(t)\rangle}$, is used.

We shall assume that $p(x, t)$ belongs to the Banach space $\Lambda_{\langle\alpha(t)\rangle}(C[0, l])$. This space is denoted by $C\Lambda_{\langle\alpha(t)\rangle}$ in [1, 4].

As in [4], we reduce the boundary value problem (1), (2) to its equivalent Volterra type integral equation

$$\left(Q(P)V - \frac{1}{P} I\right) y = \frac{1}{P} M^{-1}(P)(I+V)p \tag{4}$$

Here $M(P)$ is the differential operator generated by the differential expression

$$l_0[y] \equiv \left(-\frac{d^2}{dx^2}\left(EI(x)\frac{d^2}{dx^2} + PI\right)\right) y$$

and the boundary conditions (2), $Q(P)$ is a Fredholm operator of the form

$$(Q(P)g)(x) = \int_0^l \frac{\partial^2 Q_0(x, \xi, P)}{\partial \xi^2} g(\xi)\, d\xi \tag{5}$$

acting in $C[0, l]$, $Q_0(x, \xi, P)$ is the Green's function of the operator $M(P)$, and V is the Volterra operator (3) acting in $\Lambda_{\langle\alpha(t)\rangle}$.

A close connection exists between the Euler stability with weight $\alpha(t)$ of the boundary value problem (1), (2) and the spectrum of the operator $Q(P)V$ in the space $\Lambda_{\langle\alpha(t)\rangle}(C[0, l])$. The following lemma establishes this connection.

Lemma. In order for the boundary value problem (1), (2) to be Euler stable with weight $\alpha(t)$, it is necessary and sufficient that $1/P$ be a regular point of the operator $Q(P)V$ in the space $\Lambda_{\langle\alpha(t)\rangle}(C[0, l])$.

Proof. Sufficiency follows from (4). In order to prove the necessity, we use the theorem for multiplying spectra according to which

$$\sigma(Q(P)V) = \bigcup_i \bigcup_{\lambda\in\sigma(V)} \frac{\lambda}{P_i - P} \tag{6}$$

where $\sigma(Q(P)V)$ is the spectrum of the operator $Q(P)V$ in $\Lambda_{\langle\alpha(t)\rangle}(C[0, l])$.

If $1/P \in \sigma(Q(P)V)$, then according to (6) there is a $\lambda_0 \in \sigma(V)$ and a P_{i_0} such that

$$1/P = \lambda_0/(P_{i_0} - P)$$

This means

$$(P_{i_0} - P)/P \in \sigma(V)$$

Further, let us repeat the reasoning contained in the proof of Lemma 2 in [4] but replacing $\Lambda_{\langle e^{-\theta t}\rangle}$ by $\Lambda_{\langle\alpha(t)\rangle}$, which concludes the proof of the lemma.

This lemma and Theorem 1 from [1] permit a proof of the following theorem.

Theorem: Let $K(t, \tau) = K_0(t, \tau) + K_1(t, \tau)$, where each term satisfies conditions (1) — (3) of Theorem 1 in [1] and

$$\lim_{s\to\infty} \sup_{t\geqslant s} \int_s^t |K_1(t, \tau)| \frac{\alpha(t)}{\alpha(\tau)} d\tau = 0 \tag{7}$$

1) If $K_0(t, \tau) \geqslant 0$ in the domain $0 \leqslant \tau \leqslant t < \infty$, then the critical value of the force P is given by the formula

$$P_{\langle\alpha(t)\rangle} = P_\varepsilon/(1 + T_{k_0}) \tag{8}$$

where P_ε is the critical Euler force corresponding to (1), (2) (let us recall that $P_\varepsilon = P_1$ (see [4]).

2) If

$$\alpha(0) = 1, \quad \alpha(t+\tau) \leqslant \alpha(t)\alpha(\tau), \quad K_0(t, \tau) = K_0(t - \tau)$$

$$\theta = \lim_{t\to\infty} \frac{\ln \alpha(t)}{-t} < \infty, \quad \int_0^\infty |K_0(t)|\,\alpha(t)\,dt < \infty$$

then the following estimate holds for the critical value of the force P

$$P^{\langle\alpha(t)\rangle} \geqslant P_\varepsilon / (1+\varkappa_0), \quad (\varkappa_0 = \max_{\operatorname{Re} w \geqslant \theta} \operatorname{Re} k_0(w) \quad \text{for} \quad \operatorname{Im} k_0(w) = 0) \tag{9}$$

where $k_0(w)$ is the Laplace transform of $K_0(t)$.

Equality in the estimate (9) is achieved for $\alpha(t) = e^{-\theta t}$.

3) For subcritical values of the force P the limit deflection $(L_\alpha y)(x)$ is the solution of the boundary value problem obtained from (1),(2) with the replacements: y by $L_\alpha y$, p by $L_\alpha p$, and the Volterra operator V in (3) by the operator of multiplication by a constant T_{k_0}.

Proof. Let us start with the assertion (1). According to (6) and Theorem 1 from [1], the spectrum radius of the operator $Q(P)\,V$ in $\Lambda_{\langle\alpha(t)\rangle}(C[0, l])$ equals $T_{k_0}/(P_\varepsilon - P)$. Hence, for $1/P > T_{k_0}/(P_\varepsilon - P)$ Euler stability with the weight $\alpha(t)$, holds. If $1/P = T_{k_0}/(P_\varepsilon - P)$, then $1/P \in \sigma_{\langle\alpha(t)\rangle}(Q(P)\,V)$ and therefore (see the lemma), stability does not occur. Hence, (8) follows.

Let us prove (2). There results from Theorem 1 in [1] and the corollary to Theorem 1 in [6] that the transform of the half-plane $\operatorname{Re} w \geqslant \theta$ given by the function $k_0(w)$ covers the spectrum of the operator V in (3) in $\Lambda_{\langle\alpha(t)\rangle}$. Hence, and also from (6), it follows that if

$$1/P > \varkappa_0 / (P_\varepsilon - P) \tag{10}$$

then $1/P$ is a regular point of the operator $Q(P)\,V$ in $\Lambda_{\langle\alpha(t)\rangle}(C[0, l])$. This means that Euler stability with weight $\alpha(t)$ holds for $P < P_\varepsilon/(1+\varkappa_0)$. The estimate (9) is hence obtained.

Let us examine the case $\alpha(t) = e^{-\theta t}$ In this case

$$\varkappa_0 \in \sigma_{\langle e^{-\theta t}\rangle}(V), \quad \varkappa_0/(P_\varepsilon - P) \in \sigma_{\langle e^{-\theta t}\rangle}(Q(P)\,V)$$

Hence, if the "greater than" sign is replaced by the "equals" sign in [10), then according to the lemma, stability will not occur and the right side of the inequality (9) will yield an expression for the critical force $P^{\langle e^{-\theta t}\rangle}$.

The assertion (3), which is a direct corollary to Lemma 1 from [4], the lemma of this paper, and the easily provable equality $T_k = T_{k_0}$, affords a foundation for a method of computing the creep strength modulus under more general conditions than in [4].

REFERENCES

1. Gol'dengershel', E. I., Spectrum of a Volterra operator on a half-axis and Euler stability of viscoelastic rods. Dokl. Akad. Nauk SSSR, Vol. 217, № 4, 1974.
2. Landau, L. D. and Lifshits, E. M., Theory of Elasticity. (English translation), Pergamon Press, Book № 13128, 1959.
3. Rabotnov, Iu. N., Creep of Structural Elements. "Nauka", Moscow, 1966.
4. Gol'dengershel', E. I., On the Euler stability of a viscoelastic rod. PMM Vol. 38, № 1, 1973.
5. Andreichikov, I. P. and Iudovich, V. I., On the stability of viscoelastic rods. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, № 2, 1974.
6. Gol'dengershel', E. I., Spectrum of a Volterra operator on a half-axis and Paley-Wiener type Tauberian theorems. Dokl. Akad. Nauk SSSR, Vol. 129, № 5, 1959.
7. Gol'dengershel', E. I., On the Euler stability of a viscoelastic rod. Dokl. Akad. Nauk SSSR, Vol. 207, № 2, 1972.
8. Distefano, J. N., Creep buckling of slender columns. J. Struct. Div. Proc. Amer. Soc. Civil Engrs., Vol. 91, № 3, Pt. 1, pp. 127-149, 1965.

Translated by M. D. F.